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The Degree of Copositive Approximation

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1. INTRODUCTION

Numerous articles have been written recently on the notions of comonotone and copositive approximation: see [2–6].

We say that two functions f *and* g *are copositive on an interval* $[a, b]$ *if* $f(x)\, g(x) \geqslant 0$ *for all* x *in* $[a, b]$. Let Π_n denote the set of algebraic polynomials of degree less than or equal to n and let $\|\ \|$ be the uniform norm on $[a, b]$. Given a continuous function f on $[a, b]$ we define the *degree of copositive approximation* $\bar{E}_n(f)$ as $\inf\{\|f - p\| \mid p \in \Pi_n$ and p copositive with $f\}$. The *degree of approximation* to f is

$$E_n(f) = \inf\{\|f - p\| \mid p \in \Pi_n\}.$$

Passow and Raymon in [6] state the following theorem. See [6] for the definitions of the terms.

THEOREM A. *If* $f \in C[a, b]$ *is proper piecewise monotone with nonvanishing peaks then there is a constant* d *depending on* f *but not on* n *such that for* n *sufficiently large*

$$\bar{E}_n(f) \leqslant d w(f; 1/n). \tag{1}$$

(w is the modulus of continuity of f on $[a, b]$.)

The main theorem in this paper weakens the condition requiring f to be proper piecewise monotone and gives stronger estimates in many cases.

2. THE MAIN THEOREMS

Let f be continuous on $[-1, +1]$ and assume that there are only finitely many points $y_0 < y_1 < \cdots < y_k$ in $(-1, +1)$ at which f changes sign. Assume

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that there are numbers $\epsilon > 0$ and $\delta > 0$ so that the $k + 1$ intervals

$$I_i = [y_i - \epsilon, y_i + \epsilon], \qquad i = 0, 1, \dots, k$$

are nonoverlapping and contained in $[-1, +1]$ and so that

$$|f(x) - f(y)| \geqslant \delta \, | x - y | \tag{2}$$

whenever x and y are in the same I_i, $i = 0, 1, \dots, k$.

We will call a function f with these properties a *properly alternating function.*

THEOREM 1. *Let f be a continuous properly alternating function on $[-1, +1]$ with modulus of continuity w. Then there is a constant C depending on f but independent of n such that for n sufficiently large*

$$\bar{E}_n(f) \leqslant C\left(w\left(\frac{6E_n(f)}{\delta}\right) + E_n(f)\right). \tag{3}$$

(δ above and ϵ below are as in the definition of properly alternating function).

Proof. Let $m = \frac{1}{4}\min[\epsilon, \min_j (| y_{j+1} - y_j - 2\epsilon |)] > 0$. Let $J_i = [y_i - m, y_i + m]$ for $i = 0, 1, \dots, k$.
Let q_n be the polynomial of best approximation from Π_n to f on $[-1, +1]$. Observe that from the definition of a properly alternating function we have that f is either strictly increasing on I_i or strictly decreasing on I_i depending on how the sign changes at y_i.

If f is increasing on I_i then for any $x > y$ in J_i we have from (2)

$$\begin{aligned} q_n(x) - q_n(y) &\geqslant f(x) - f(y) - 2E_n(f) \\ &\geqslant | x - y | \, \delta - 2E_n(f) \\ &\geqslant 4E_n(f) \qquad \text{if } | x - y | \geqslant 6E_n(f)/\delta \end{aligned} \tag{4}$$

If f is decreasing on I_i then for $x > y$ in J_i we have from (2)

$$\begin{aligned} q_n(x) - q_n(y) &\leqslant 2E_n(f) + f(x) - f(y) \\ &\leqslant 2E_n(f) - \delta \, | x - y | \\ &\leqslant -4E_n(f) \qquad \text{if } | x - y | \geqslant 6E_n(f)/\delta \end{aligned} \tag{5}$$

Now define

$$\alpha_n(x) = \left(1 - \frac{3E_n(f)}{\delta}\right)x - \frac{3E_n(f)}{\delta}$$

and

$$\beta_n(x) = \alpha_n(x) + \frac{6E_n(f)}{\delta}. \tag{6}$$

It is easy to see that for all x in $[-1, +1]$

$$-1 \leqslant \alpha_n(x) \leqslant x \leqslant \beta_n(x) \leqslant 1. \tag{7}$$

Now define

$$s_n(x) = \frac{\delta}{6E_n(f)} \int_{\alpha_n(x)}^{\beta_n(x)} q_n(t)\, dt \tag{8}$$

and choose N_1 so that $6E_n(f)/\delta < \min(1, m)$ for $n \geqslant N_1$.
This together with (6) and (7) show that if $x \in J_i$ then both $\alpha_n(x)$ and $\beta_n(x)$ are in I_i for $i = 0, 1,..., k$.
Observe that

$$s_n'(x) = \frac{\delta}{6E_n(f)} [q_n(\beta_n(x)) - q_n(\alpha_n(x))] \left(1 - \frac{3E_n(f)}{\delta}\right).$$

Also for $n \geqslant N_1$ we have

$$1 - 3E_n(f)/\delta \geqslant \tfrac{1}{2}.$$

These facts together with (4) and (5) show that for $n \geqslant N_1$, if x is in J_i we have

$$s_n'(x) \geqslant \delta/3 \tag{9}$$

if f is increasing on J_i and

$$s_n'(x) \leqslant -\delta/3 \tag{10}$$

if f is decreasing on J_i. Now choose $r_{k,n} \in \Pi_k$ so that $r_{k,n}(y_i) = -s_n(y_i)$ for $i = 0, 1,..., k$. Let $N_2 = \max(N_1, k)$. Then for $n \geqslant N_2$ we still have (9) and (10). Now define

$$t_n(x) = s_n(x) + r_{k,n}(x). \tag{11}$$

Then $t_n \in \Pi_n$ for $n \geqslant N_2$.
Now observe that

$$f(x) = \frac{\delta}{6E_n(f)} \int_{\alpha_n(x)}^{\beta_n(x)} f(x)\, dt.$$

Hence

$$f(x) - s_n(x)$$
$$= \frac{\delta}{6E_n(f)} \int_{\alpha_n(x)}^{\beta_n(x)} (f(x) - f(t))\, dt + \frac{\delta}{6E_n(f)} \int_{\alpha_n(x)}^{\beta_n(x)} (f(t) - q_n(t))\, dt.$$

Thus from (6) and (7)

$$|f(x) - s_n(x)| \leqslant w(6E_n(f)/\delta) + E_n(f). \tag{12}$$

Let $D_n = w(6E_n(f)/\delta) + E_n(f)$.
Note that

$$\begin{aligned} |r_{k,n}(y_i)| &= |f(y_i) - s_n(y_i)| \\ &\leqslant D_n, \qquad i = 0, 1,..., k. \end{aligned}$$

From the Lagrange interpolation formula it is clear that there is a constant B_k depending only on y_0 ,..., y_k such that

$$|r_{k,n}(x)| \leqslant B_k D_n \qquad \text{on } [-1, +1] \tag{13}$$

for all n.
By the inequality of Markov (see [1])

$$|r'_{k,n}(x)| \leqslant k^2 B_k D_n \tag{14}$$

on $[-1, +1]$ for all n.
Now choose $N_3 \geqslant N_2$ so that $n \geqslant N_3$ gives

$$k^2 B_k D_n < \delta/6. \tag{15}$$

Then on J_i we see that for $n \geqslant N_3$ the sign of $t_n'(x)$ is the same as the sign of $s_n'(x)$ since from (9), (10), (14), (15) we have

$$|s_n'(x)| - |r'_{k,n}(x)| \geqslant \delta/6. \tag{16}$$

Moreover, we have for $x \in [-1, +1]$ using (11), (12), and (13),

$$|f(x) - t_n(x)| \leqslant (1 + B_k)\, D_n\,. \tag{17}$$

Moreover $t_n(y_i) = f(y_i) = 0$ for $i = 0, 1,..., k$.
This guarantees that f *and* t_n *are copositive on each* J_i $i = 0, 1,..., k$. To complete the proof we will add to t_n a certain polynomial that is copositive with f on $[-1, +1]$. From (16) we have for $n \geqslant N_3$ and x in J_i

$$|t_n'(x)| \geqslant \delta/6. \tag{18}$$

Define $h_k(x) = (x - y_0)(x - y_1) \cdots (x - y_k)$. We may assume that h_k and f are copositive on $[-1, +1]$. Otherwise we take $-h_k$. Define $\mathscr{B} = [-1, +1] - \bigcup_{i=0}^{k} J_i$, and set

$$\rho = \inf\{|h_k(x)| \mid x \in \mathscr{B}\} > 0. \tag{19}$$

Form

$$p_n(x) = t_n(x) + \frac{(2 + B_k) D_n}{\rho} h_k(x) \tag{20}$$

and set $N_4 = \max(N_3, k + 1)$. Then for $n \geqslant N_4$ we have $p_n \in \Pi_n$. Let $C_k = \max\{|h_k(x)| \mid x \in [-1, +1]\}$. Then we have from (17) and (20) that

$$|f(x) - p_n(x)| \leqslant A_k D_n \tag{21}$$

where

$$A_k = 1 + B_k + \frac{(2 + B_k)}{\rho} C_k .$$

Moreover if $x \in \mathscr{B}$ we have

$$\left| \frac{(2 + B_k) D_n}{\rho} h_k(x) \right| \geqslant (2 + B_k) D_n . \tag{22}$$

Clearly if f and t_n are copositive at some x, then so are f and p_n. Hence, f and p_n are copositive on $\bigcup_{i=0}^{k} J_i$. If f and t_n are not copositive at some $\bar{x}$ then $\bar{x} \in \mathscr{B}$. Assume without loss of generality that $f(\bar{x}) \geqslant 0$. Then using (17), (20), (22) and the fact that f and h_k are copositive we have

$$\begin{aligned} p_n(\bar{x}) &= t_n(\bar{x}) + ((2 + B_k)/\rho) D_n h_k(\bar{x}) \\ &\geqslant f(\bar{x}) - |f(\bar{x}) - t_n(\bar{x})| + (2 + B_k) D_n \\ &\geqslant f(\bar{x}) - (1 + B_k) D_n + (2 + B_k) D_n \\ &= f(\bar{x}) + D_n > 0. \end{aligned}$$

Hence $f(\bar{x})$ and $p_n(\bar{x})$ have the same sign. Thus f and p_n are copositive for $n \geqslant N_4$ and the theorem is proved. ∎

It is clear that the class of functions treated in Theorem A is properly contained in the class of properly alternating functions. The following corollary emphasizes the comparison between Theorem A and Theorem 1.

COROLLARY. *If f is a properly alternating function and if $f \in \mathrm{Lip}_M 1$ on $[-1, 1]$ then there is a constant B depending on f but independent of n such that*

$$\bar{E}_n(f) \leqslant BE_n(f) \qquad \textit{for } n \textit{ sufficiently large.} \tag{23}$$

It is easy to see using the classical Jackson's theorems that (23) is better than (1).

The proof of the next theorem is contained in [4] but the theorem is not stated explicitly there since the emphasis is on comonotone approximation. In the next theorem we allow a sign change to be on an interval. That is, $f(x) = 0$ on $[a, b]$, $f(x) < 0$ on $[e, a)$, and $f(x) > 0$ on $(b, d]$. In this case we will set $C = a + b/2$ and say f changes sign at C.

THEOREM 2. *Let f have sign changes at $y_1 < \cdots < y_k$ on* $[-1, +1]$ and *assume f is continuous on $[-1, +1]$ and $f'(y_i)$ exists for $i = 1,..., k$. Define $g(x) = f(x)/\prod_{i=1}^{k} (x - y_i)$. Then g is continuous on $[-1, +1]$ and*

$$\bar{E}_n(f) \leqslant CE_{n-k}(g) \qquad \text{for } n \geqslant k \tag{24}$$

where C depends only on $y_1 ,..., y_k$.

We omit the proof since it is contained in [4] and is, in any event, easy to construct.

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